Title
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Journal
Experimental Mathematics, 24(2)

ISSN
1058-6458

Authors
Gorsky, E
Lewark, L

Publication Date
2015-04-03

DOI
10.1080/10586458.2014.963746

Peer reviewed

ON STABLE sl_3-HOMOLOGY OF TORUS KNOTS

EUGENE GORSKY AND LUKAS LEWARK

ABSTRACT. The stable Khovanov-Rozansky homology of torus knots has been conjecturally described as the Koszul homology of an explicit non-regular sequence of polynomials. We verify this conjecture against newly available computational data for sl_3-homology. Special attention is paid to torsion. In addition, explicit conjectural formulae are given for the sl_N-homology of $(3, m)$-torus knots for all N and m, which are motivated by higher categorified Jones-Wenzl projectors. Structurally similar formulae are proven for Heegard-Floer homology.

1. INTRODUCTION

Torus knots are among the simplest and best understood knots, and their invariants have been studied and computed using various mathematical and physical tools. For example, Jones and Rosso [12] computed all sl_N-quantum invariants of torus knots explicitly.

The computation of Khovanov and Khovanov-Rozansky homology of torus knots remains an important open problem in knot theory. The experimental computations of Bar-Natan, Shumakovitch, Turner [2, 3, 19, 20] and others revealed a rich structure, including nontrivial torsion, in sl_2-homology of torus knots. On the other hand, the main conjecture of [11] related the HOMFLY-PT homology of (n, m)-torus knots to finite dimensional representations of rational Cherednik algebras with parameter m/n. In the limit $m \to \infty$, this construction has been used in [10] for a conjectural description of stable sl_N-homology of torus knots. For $N = 2$, this conjecture has been verified against the experimental data for stable torus knots with up to 8 strands, and they agree both in reduced and in unreduced homology.

Conjecture 1. *([10]) Consider the triply graded algebra $\mathcal{H}_n$ with even generators $x_0, \ldots, x_{n-1}$ and odd generators $\xi_0, \ldots, \xi_{n-1}$ of degrees*

$$\deg x_k = q^{2k+2}t^{2k}, \quad \deg \xi_k = a^2q^{2k}t^{2k+1}.$$

It is equipped with the family of Koszul differentials d_N defined by the equations:

$$d_N(x_k) = 0, \quad d_N(\xi_k) = \sum_{i_1+\ldots+i_N=k} x_{i_1} \cdot \ldots \cdot x_{i_N}. \tag{1}$$

Then the stable sl_N-Khovanov-Rozansky homology of the (n, ∞)-torus knot is isomorphic to $H^*(\mathcal{H}_n, d_N)$ after the regrading $a = q^N$.

Recently the second named author developed a computer program [13, 14] that allows one to compute integral sl_3-Khovanov-Rozansky homology of any knot, in reasonable time for knots such as the $(5, 14)$- or $(6, 7)$-torus knots. The main goal of this article is to verify Conjecture 1 for $N = 3$ using the newly available experimental data, see Section 4.

In Section 2, we study Conjecture 1, proving it to be correct for the $(2, \infty)$-torus knot, and giving an explicit formula for the 3-strand case. Reduced homology is related to unreduced. We

Date: April 2, 2014.

The first author is partially supported by the grants RFBR13-01-00755, NSh-4850.2012.1. The second author is supported by the EPSRC-grant EP/K00591X/1.

show that for a prime $p > N+1$, the Koszul model for stable sl_N-homology of the p-strand torus knot has p-torsion, and give a formula for the Koszul model of sl_N-homology with $\mathbb{Z}_N$-coefficients, which has a relatively simple form. Note that while a definition of sl_N-homology with integer or $\mathbb{Z}_p$-coefficients may be expected to exist, it has so far only been given for $N=2$ and $N=3$.

For 3-strand finite torus knots $T(3,m)$, we also write down explicit conjectural formulae for the Poincaré polynomials of sl_N-homology (see Section 3). Namely, for each of the 4 standard Young tableaux T with 3 boxes we define a certain explicit rational function $\mathcal{P}_T(N)$.

Conjecture 2. *The Poincaré series of the unreduced sl_N-homology of the $(3,3k+1)$-torus knot equals (up to an overall shift of $q^{3k(N-1)-2}$):*

$$\mathcal{P}(T(3,3k+1),sl_N) = \mathcal{P}_{\ytableaushort{123}}(N) + q^{6k}t^{4k}\mathcal{P}_{\ytableaushort{12,3}}(N) + q^{6k}t^{4k}\mathcal{P}_{\ytableaushort{13,2}}(N) + q^{12k}t^{6k}\mathcal{P}_{\ytableaushort{1,2,3}}(N).$$

The Poincaré series of the unreduced sl_N-homology of the $(3,3k+2)$-torus knot equals (up to an overall shift of $q^{3k(N-1)-3}$):

$$\mathcal{P}(T(3,3k+2),sl_N) = \mathcal{P}_{\ytableaushort{123}}(N) + q^{6k}t^{4k}\mathcal{P}_{\ytableaushort{12,3}}(N) +$$
$$q^{6k+4}t^{4k+2}\mathcal{P}_{\ytableaushort{13,2}}(N) + q^{12k+4}t^{6k+2}\mathcal{P}_{\ytableaushort{1,2,3}}(N).$$

The conjecture is motivated by the work [9], where the "refined Chern-Simons invariants" [1] of (n,m)-torus knots were expressed as sums over standard Young tableaux of size n. It was conjectured in [1] that these invariants agree with the Poincaré polynomials of HOMFLY-PT homology. We show that for $n=3$ (and $n=2$) the HOMFLY-PT analogues of $\mathcal{P}_T(N)$ can be viewed as Poincaré series of certain free supercommutative algebras, and $\mathcal{P}_T(N)$ can be obtained as homology of certain differentials d_N acting on these algebras. In particular, $\mathcal{P}_{\ytableaushort{123}}(N)$ coincides with the Poincaré series of stable sl_N-homology of $T(3,\infty)$ which, by Conjecture 1, can be described as homology of d_N. This decomposition is also expected to be related to the categorification of higher Jones-Wenzl projectors [5], but we do not pursue this relation here.

Surprisingly enough, we prove that the Poincaré polynomials of the Heegaard-Floer homology of $T(3,m)$ can be written using a formula with a similar structure. The corresponding differential corresponds to d_0 in the notations of [7, 11] and to $d_{1|1}$ in the notations of [8].

ACKNOWLEDGMENTS

The authors would like to thank M. Khovanov, A. Lobb and J. Rasmussen for the useful discussions and remarks.

2. THE KOSZUL MODEL FOR STABLE sl_N-HOMOLOGY

2.1. **Koszul model.** In [10, 11] the following algebraic model for the stable homology of torus knots was proposed.

Conjecture 1. ([10]) *Consider the triply graded algebra $\mathcal{H}_n$ with even generators $x_0,\ldots,x_{n-1}$ and odd generators $\xi_0,\ldots,\xi_{n-1}$ of degrees*

$$\deg x_k = q^{2k+2}t^{2k}, \quad \deg \xi_k = a^2q^{2k}t^{2k+1}.$$

It is equipped with the family of Koszul differentials d_N defined by the equations:

$$d_N(x_k) = 0, \quad d_N(\xi_k) = \sum_{i_1+\ldots+i_N=k} x_{i_1} \cdot \ldots \cdot x_{i_N}. \tag{1}$$

Then the stable sl_N-Khovanov-Rozansky homology of the (n, ∞)-torus knot is isomorphic to $H^*(\mathcal{H}_n, d_N)$ after the regrading $a = q^N$.

Example 3. *For $n = 1$ one has a single even generator x_0 and a single odd generator ξ_0, and the differential has the form $d_N(\xi_0) = x_0^N$. The homology has dimension N and is spanned by $1, x_0, \ldots, x_0^{N-1}$, so it is indeed isomorphic to the sl_N-homology of the unknot.*

We denote the homology of d_N as $H_n^{\text{alg}}(sl_N)$. After setting $a = q^N$ the variables are graded as follows:

$$\deg x_k = q^{2k+2}t^{2k}, \quad \deg \xi_k = q^{2N+2k}t^{2k+1},$$

so that d_N preserves the q-grading and decreases the t-grading by 1.

2.2. **Generators.** Consider the generating functions $x(\tau) = \sum_{k=0}^{\infty} x_k\tau^k$ and $\xi(\tau) = \sum_{k=0}^{\infty} \xi_k\tau^k$. The equation (1) can be rewritten as

$$d_N(x(\tau)) = 0, \quad d_N(\xi(\tau)) = x(\tau)^N \mod \tau^n. \tag{2}$$

Remark that

$$d_N(\dot{\xi}(\tau)) = N\dot{x}(\tau)x(\tau)^{N-1} \mod \tau^n. \tag{3}$$

Define

$$\mu(\tau) = \sum_{k=1}^{\infty} \mu_k\tau^{k-1} := N\dot{x}(\tau)\xi(\tau) - x(\tau)\dot{\xi}(\tau),$$

so $\mu_k = \sum_{i+j=k}(Ni - j)x_i\xi_j$. It follows from (2) and (3) that $d_N(\mu(\tau)) = 0$, so $d_N(\mu_k) = 0$ for all k. The degree of μ_k equals $q^{2N+2k+2}t^{2k+1}$.

Conjecture 4. *The homology of d_N is generated as an algebra by x_k and μ_k.*

2.3. **Reduced homology.** The reduced homology is defined as a quotient of the unreduced homology by the homology of the unknot, i.e. by the equation $x_0 = 0$. It turns out that the reduced and unreduced Koszul homology are tightly related.

Theorem 5. *We have the following isomorphism:*

$$H_n^{\text{alg,red}}(sl_N) \simeq \mathbb{Z}[\xi_1, \ldots, \xi_{N-1}, x_{n-N+1}, \ldots, x_n] \otimes H_{n-N}^{\text{alg}}(sl_N).$$

Proof. In reduced homology we set $x_0 = 0$, so $d_N(\xi_i) = 0$ for $i < N$. Furthermore, $d_N^{\text{red}}(\xi_i)$ coincides with $d_N(\xi_{i-N})$ where all x_i are replaced by x_{i-1}. For example, $d_N^{\text{red}}(\xi_N) = x_1^N$. □

Note that the isomorphism does not preserve q- and t-gradings, but it is easy to reconstruct the change of gradings transforming x_i to x_{i+1} and ξ_i to ξ_{i+N}.

2.4. $\mathbb{Z}_N$**-homology for prime** N**.** In this subsection we give a simple description of sl_N-homology with $\mathbb{Z}_N$-coefficients, assuming for simplicity that N is a prime number.

Theorem 6. *Suppose that N is prime. Then the algebraic Poincaré series has the form:*

$$\mathcal{P}_n^{\text{alg}}(sl_N, \mathbb{Z}_N) = \prod_{k=0}^{n-1} \frac{1 + q^{2N+2k}t^{2k+1}}{1 - q^{2k+2}t^{2k}} \prod_{i=0}^{\lfloor \frac{n-1}{N} \rfloor} \frac{1 - q^{2N+2iN}t^{2iN}}{1 + q^{2N+2iN}t^{2iN+1}}.$$

Proof. If N is prime, then

$$d_N(\xi(\tau)) = x(\tau)^N \equiv \sum x_i^N \tau^{iN} \mod N.$$

Therefore

$$d_N(\xi_k) \equiv \begin{cases} 0 \mod N, & k \neq Ni \\ x_i^N \mod N, & k = Ni. \end{cases}$$

The homology is generated by x_i modulo relations $x_i^N = 0$ and by ξ_k for k not divisible by N. □

See Table 1 for an illustration of the theorem.

2.5. **Torsion.** The computations of Khovanov homology showed the presence of large and complicated torsion. The first computations in sl_3-knot homology indeed confirmed the existence of torsion as well. It has been shown in [10] that the algebraic model is compatible at least with some part of this torsion, the following result generalizes this check to sl_N-homology.

Theorem 7. *Let $p > N+1$ be a prime number, then the algebraic homology $H_p^{\mathrm{alg}}(sl_N, \mathbb{Z})$ has p-torsion in bidegree $q^{2p+2N+2}t^{2p+1}$.*

Proof. Recall that for $\mu_p = \sum_{i=0}^{p}(Ni-p+i)x_i\xi_{p-i}$ one has $d_N(\mu_p) = 0$. This generator is not present in $\mathcal{H}_p$, since x_p and ξ_p are not present in the Koszul complex. Consider the element

$$t_p := \sum_{i=1}^{p-1}(Ni-p+i)x_i\xi_{p-i} = \mu_p - Npx_p\xi_0 + px_0\xi_p,$$

which is present in $\mathcal{H}_p$. We have

$$d_N(t_p) = p(x_0 d_N(\xi_p) - Nx_p x_0^N),$$

so $d_N(t_p)$ vanishes modulo p. Since t_p has a term $(N-p+1)x_1\xi_{p-1}$ that does not vanish modulo p, the dimension of the kernel of d_N over $\mathbb{Z}_p$ is bigger than its rank over $\mathbb{Z}$. □

Example 8. *For $N = 3$ and $p = 5$ the computations of* [14] *show $\mathbb{Z}_5$-torsion in bidegree $q^{18}t^{11}$ of the homology of the $(5,m)$-torus knot for $6 \leq m \leq 14$. It is plausible that it survives in the stable limit and hence agrees with the torsion generator t_5. See Table 2 for the case $m = 9$.*

Indeed, the torsion classes described in Theorem 7 (and their products with x's and μ's) do not cover all stable torsion.

Example 9. *Consider the following class in $\mathcal{H}_n$ (defined for $n \geq 5$):*

$$A := 4x_0\xi_1\xi_4 - 8x_1\xi_0\xi_4 + 8x_4\xi_0\xi_1 - 2x_0\xi_2\xi_3 - 2x_2\xi_1\xi_2 - 4x_3\xi_0\xi_2 + x_1\xi_1\xi_3 + 4x_2\xi_0\xi_3.$$

One can check that $d_2(A) = 10x_1x_3(2x_1\xi_0 - x_0\xi_1)$, so A represents a class in $H_n^{\mathrm{alg}}(sl_2, \mathbb{Z}_5)$. On the other hand, one can check (say, from Conjecture 4) that A cannot be lifted to a class in $H_n^{\mathrm{alg}}(sl_2, \mathbb{Z})$, hence $H_n^{\mathrm{alg}}(sl_2, \mathbb{Z})$ has 5-torsion in bidegree $q^{20}t^{12}$ for all $n \geq 5$. This torsion can be seen in the data of [3]*, e.g. for the $(5,6)$-, $(6,7)$-, $(7,8)$- and $(8,9)$- torus knots.*

Example 10. *Consider the following class in $\mathcal{H}_6$:*

$$B := 5x_1^2\xi_5 - 5x_1x_5\xi_1 - 10x_1x_4\xi_2 + 16x_2x_4\xi_1 - 15x_1x_3\xi_3 + 15x_3^2\xi_1 + 3x_2^2\xi_3 - 3x_2x_3\xi_2 - 6x_1x_2\xi_4.$$

One can check that $d_3(B) = -105x_0x_1^2x_2x_3$, so B represents a class in $H_6^{\mathrm{alg}}(sl_3, \mathbb{Z}_7)$. Moreover, B cannot be lifted to a class in $H_6^{\mathrm{alg}}(sl_3, \mathbb{Z})$, so $H_6^{\mathrm{alg}}(sl_3, \mathbb{Z})$ has 7-torsion in bidegree $q^{24}t^{15}$. This is also confirmed by the experimental data [14]*, i.e. by the computed homology of the*

$\ddagger$ \ t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
6				1		1		1	1		1		1																	
4	1		1	1	1	2		3	1	2	1	1	2		1		1													
2	1		1	1	2	2	1	3	2	4	1	4	3	3	3	1	4	1	2	1	1	1								
0	1		1		2	1	2	2	3	5	2	7	3	8	4	6	7	4	6	2	5	2	2	1	1	1				
-2					1		2		4	3	4	6	5	10	5	11	7	11	8	8	9	5	7	3	5	2	2	1	1	
-4							1		3	1	4	3	6	7	6	11	8	14	8	13	10	11	9	8	8	5	4	2	2	1
-6									1		2		5	2	6	6	7	11	7	13	9	13	9	11	8	7	5	4	3	1
-8													2		4	1	6	4	6	7	7	10	6	9	6	7	4	3	2	1
-10															1		2		4	1	5	3	4	4	3	4	2	2	1	
-12																			1		2		2		2	1	1			

TABLE 1. The unreduced sl_3-homology of $T(5,9)$ with $\mathbb{Z}_3$-coefficients, cf. Theorem 6. The top row indicates the homological degree t, and the left-most column $\ddagger = q - 2t$, where q is the quantum degree. For simplicity, we follow the grading convention (which is different from the Khovanov and Rozansky's original one) that leads to e.g. the reduced sl_3-homology of the trefoil with positive crossings having Poincaré polynomial $q^4 + t^2q^8 + t^3q^{12}$. Note also that the stable part of an actual (n,m)-torus knot is shifted in the quantum degree by $2(nm - n - m)$ relative to the homology of the (n,∞)-torus knot. Each cell shows the dimension of homology at the corresponding degree. The first divergence from the stable model (at homological degree 16) is indicated by a doubled vertical line.

$q \backslash t$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
6				1		1			1		1		1																
4	1			1		2		2		1	1		2		1														
2	1		1		1	1		3		4		2	2		3		3		1	1		1							
0	1		1		2		2	1	1	4		6		5	$_{\mathbf{1}}1$	1	5		5		3	1		1		1			
−2					1		2		3	1	1	4		8		8	$_{\mathbf{1}}1$	4	$_{\mathbf{1}}5$		7		5	$_{\mathbf{1}}$	3	1	1	1	
−4							1		3		4	$\boxed{_{\mathbf{1}}}$	4	$_{\mathbf{1}}3$	1	9		11	$_{\mathbf{1}}$	8	$_{\mathbf{2}}2$	3	$_{\mathbf{1}}5$		6		3		1
−6									1		2		5		5	$_{\mathbf{1}}2$	3	$_{\mathbf{1}}6$		10		10	$_{\mathbf{2}}$	7	$_{\mathbf{2}}1$	3	$_{\mathbf{1}}2$	1	1
−8													2		4		6	$_{\mathbf{1}}$	5	$_{\mathbf{2}}2$	3	$_{\mathbf{1}}6$		7		5		2	
−10															1		2		4		5	$_{\mathbf{1}}$	4	$_{\mathbf{2}}$	2	$_{\mathbf{1}}1$	1	1	
−12																			1		2		2		2		1		

TABLE 2. The unreduced sl_3-homology of $T(5,9)$. See the caption of Table 1 for detailed explanations. The $\mathbb{Z}_5$-torsion is printed at the lower left corner of each cell in bold and green; for the sake of legibility, $\mathbb{Z}_3$-torsion is not printed, but cf. Table 1. The only other torsion is $\mathbb{Z}_2$-torsion $(t^{23}q^{36} + t^{24}q^{42})$, which is not printed either, and may be expected to be unstable. Note the presence of $\mathbb{Z}_5$-torsion at $t^{11}q^{18}$ (indicated by a box), in accordance with Theorem 7.

$(6,7)$-torus knot and $(6,8)$-torus link. Note that $B \equiv x_2\mu_5 \mod 5$, so it does not contribute to the 5-torsion.

Remark 11. *Przytycki and Sazdanović conjectured in* [16, Conjecture 6.1] *that the closure of an n-strand braid cannot have $\mathbb{Z}_p$-torsion in Khovanov homology for a prime $p > n$. Example 10 shows that a naive generalization of this conjecture ("the closure of an n-strand braid cannot have $\mathbb{Z}_p$-torsion in sl_N-homology for a prime $p > \max(n, N)$") does not hold for sl_3-homology, since the $(6,7)$-torus knot and the $(6,8)$-torus link already have 7-torsion.*

2.6. $(2,\infty)$ **case.** For $n = 2$ we have two even generators x_0, x_1 and two odd generators ξ_0, ξ_1, with the differential

$$d_N(\xi_0) = x_0^N, \quad d_N(\xi_1) = Nx_0^{N-1}x_1.$$

There is a generator $\mu_1 = Nx_1\xi_0 - x_0\xi_1$ of degree $q^{2N+4}t^3$, the homology is generated by x_0, x_1 and μ_1 modulo relations

$$x_0^N = 0, \quad Nx_0^{N-1}x_1 = 0, \quad x_0^{N-1}\mu_1 = 0.$$

The monomial basis in $\mathbb{Q}$-homology has the form

$$x_0^{N-1},\ x_0^i x_1^a \mu_1^b,\ i < N-1, b \leq 1,$$

so the Poincaré series has the form

$$\mathcal{P}_2(q,t) = q^{2N-2} + \frac{(1-q^{2N-2})(1+q^{2N+4}t^3)}{(1-q^2)(1-q^4t^2)} \tag{4}$$

$$= \frac{1 - q^{2N} - q^{2N+2}t^2 + q^{2N+4}t^2 + q^{2N+4}t^3 - q^{4N+2}t^3}{(1-q^2)(1-q^4t^2)}.$$

The actual sl_N-Khovanov-Rozansky homology was computed for all N by Cautis in [4, Corollary 10.2]. Since the two answers coincide, we obtain the following result.

Theorem 12. *Conjecture 1 is true for $n = 2$ and all N.*

2.7. $(3,\infty)$ **case.** For $n = 3$ we have new generators x_2 and ξ_2 in $\mathcal{H}_3$, and the differential looks like

$$d_N(\xi_2) = Nx_0^{N-1}x_2 + \binom{N}{2}x_0^{N-2}x_1^2.$$

We have a new homology generator

$$\mu_2 = 2Nx_2\xi_0 + (N-1)x_1\xi_1 - 2x_0\xi_2, \qquad \deg \mu_2 = q^{2N+6}t^3$$

and one can explicitly check the identity $d_N(\mu_2) = 0$.

In zeroth Koszul homology we get

$$\mathbb{Q}[x_0, x_1, x_2]/\left(x_0^N, Nx_0^{N-1}x_1, Nx_0^{N-1}x_2 + \binom{N}{2}x_0^{N-2}x_1^2\right).$$

If we multiply the third term by x_1 and subtract x_2 times the second one, we get $x_0^{N-2}x_1^3 = 0$. The monomial basis consists of all monomials not divisible by $x_0^N, x_0^{N-1}x_1, x_0^{N-1}x_2$ and $x_0^{N-2}x_1^3$, and the Poincaré series equals

$$\mathcal{P}_3^{a=0}(q,t) = q^{2N-2} + \frac{1-q^{2N-2}}{(1-q^2)(1-q^4t^2)(1-q^6t^4)} - \frac{q^{2N+8}t^6}{(1-q^4t^2)(1-q^8t^6)}$$

$$= \frac{1 - q^{2N} - q^{2N+2}t^2 - q^{2N+4}t^4 + q^{2N+4}t^2 + q^{2N+6}t^4}{(1-q^2)(1-q^4t^2)(1-q^6t^4)}.$$

Let us describe the relations between μ's. We have

$$\begin{aligned} d_N(\xi_0\xi_1) &= x_0^{N-1}\mu_1, \\ d_N(2\xi_0\xi_2) &= 2x_0^N\xi_2 - \left(2Nx_0^{N-1}x_2 + N(N-1)x_0^{N-2}x_1^2\right)\xi_0 \\ &= -x_0^{N-1}\mu_2 - (N-1)x_0^{N-2}x_1\mu_1, \\ d_N(2\xi_1\xi_2) &= 2Nx_0^{N-1}x_1\xi_2 - \left(2Nx_0^{N-1}x_2 + N(N-1)x_0^{N-2}x_1^2\right)\xi_1 \\ &= -Nx_0^{N-2}x_1\mu_2 + 2Nx_0^{N-2}x_2\mu_1. \end{aligned}$$

Therefore the monomials cannot be divisible by $x_0^{N-1}\mu_1$, $x_0^{N-1}\mu_2$ and $x_0^{N-2}x_1\mu_2$, and the Poincaré series equals

$$\mathcal{P}_3^{a=1}(q,t) = \frac{(1-q^{2N-2})(q^{2N+4}t^3 + q^{2N+6}t^5) - q^{4N+6}t^7(1-q^4)}{(1-q^2)(1-q^4t^2)(1-q^6t^4)}.$$

Finally, the homology at level 2 is generated by $\mu_1\mu_2$ and the unique relation has the form:

$$d_N(2\xi_0\xi_1\xi_2) = x_0^{N-2}\mu_1\mu_2,$$

hence

$$\mathcal{P}_3^{a=2}(q,t) = \frac{q^{4N+10}t^8(1-q^{2N-4})}{(1-q^2)(1-q^4t^2)(1-q^6t^4)}.$$

We obtain the following result:

Theorem 13. *The Poincaré series for the rational homology $H_3^{\mathrm{alg}}(sl_N,\mathbb{Q})$ is given by the formula*

$$\begin{aligned} \mathcal{P}_3(sl_N,\mathbb{Q})(1-q^2)(1-q^4t^2)(1-q^6t^4) &= 1 - q^{2N} - q^{2N+2}t^2 + q^{2N+4}(t^2+t^3-t^4) \\ &+ q^{2N+6}(t^4+t^5) - q^{4N+2}t^3 - q^{4N+4}t^5 - q^{4N+6}t^7 + q^{4N+10}(t^7+t^8) - q^{6N+6}t^8. \end{aligned}$$

3. Projectors and sl_N-homology of $(3,m)$-torus knots

In this section we give a conjectural formula for sl_N-homology of $(3,m)$-torus knots for all N and m. The construction is motivated by the study of higher categorified Jones-Wenzl projectors, introduced by Cooper and Hogancamp [5], although we do not pursue this analogy here. These projectors are labeled by the standard Young tableaux (SYT) and categorify the projectors onto irreducible summands in the regular representation of the Hecke algebra. By a theorem of Rozansky [18] (see also [4, 17]), the homology of the categorified projector corresponding to the symmetric representation of H_n coincides with the stable homology of the (n,∞)-torus knot.

3.1. **Decomposition of the homology.** We expect the Poincaré polynomials of homology of three-strand torus knots to be defined by the following conjecture.

Conjecture 2. *The Poincaré series of the unreduced sl_N-homology of the $(3,3k+1)$-torus knot equals (up to an overall shift of $q^{3k(N-1)-2}$):*

$$\mathcal{P}(T(3,3k+1),sl_N) = \mathcal{P}_{\begin{smallmatrix}1&2&3\end{smallmatrix}}(N) + q^{6k}t^{4k}\mathcal{P}_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}(N) + q^{6k}t^{4k}\mathcal{P}_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}(N) + q^{12k}t^{6k}\mathcal{P}_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}(N).$$

The Poincaré series of the unreduced sl_N-homology of the $(3, 3k+2)$-torus knot equals (up to an overall shift of $q^{3k(N-1)-3}$):

$$\mathcal{P}(T(3,3k+2), sl_N) = \mathcal{P}_{\boxed{1}\boxed{2}\boxed{3}}(N) + q^{6k}t^{4k}\mathcal{P}_{\begin{smallmatrix}\boxed{1}&\boxed{2}\\ \boxed{3}&\end{smallmatrix}}(N)+$$

$$q^{6k+4}t^{4k+2}\mathcal{P}_{\begin{smallmatrix}\boxed{1}&\boxed{3}\\ \boxed{2}&\end{smallmatrix}}(N) + q^{12k+4}t^{6k+2}\mathcal{P}_{\begin{smallmatrix}\boxed{1}\\ \boxed{2}\\ \boxed{3}\end{smallmatrix}}(N).$$

The conjecture is expected to hold both for reduced and unreduced sl_N-and HOMFLY-PT homology of $(3, m)$-torus knots, and we prove below that a similar decomposition holds for the Heegaard-Floer homology, too. For $N = 2$, the answer agrees with the computations of Turner [20]. For $N = 3$, we checked the answer on the computer both for large ($k = 28$) and small torus knots and found a perfect agreement with the experimental data.

The existence of such decompositions was first conjectured by Oblomkov and Rasmussen [15], but, to the knowledge of the authors, the explicit formulae for $\mathcal{P}_T$ have not been written down before.

3.2. **Symmetric projector, two strands.** The unreduced HOMFLY-PT homology of the symmetric projector is a free algebra with generators x_0, x_1, ξ_0, ξ_1 with the degrees:

$$\deg(x_0) = q^2, \quad \deg(x_1) = q^4t^2, \quad \deg(\xi_0) = a^2t, \quad \deg(\xi_1) = a^2q^2t^3.$$

Therefore the Poincaré series of this homology equals:

$$\mathcal{P}_{\boxed{1}\boxed{2}} = \frac{(1+a^2t)(1+a^2q^2t^3)}{(1-q^2)(1-q^4t^2)}.$$

The reduced HOMFLY-PT homology is generated by x_1 and ξ_1, and the Poincaré series equals:

$$\mathcal{P}^{\mathrm{red}}_{\boxed{1}\boxed{2}} = \frac{(1+a^2q^2t^3)}{(1-q^4t^2)}.$$

According to Conjecture 1, the sl_N-homology can be described as the homology of the differential d_N after regrading $a = q^N$. The unreduced sl_N-homology of the symmetric projector are then given by (4):

$$\mathcal{P}^{d_N}_{\boxed{1}\boxed{2}} = \frac{1 - q^{2N} - q^{2N+2}t^2 + q^{2N+4}t^2 + q^{2N+4}t^3 - q^{4N+2}t^3}{(1-q^2)(1-q^4t^2)}.$$

Remark that the Euler characteristic of this projector is equal to the S^2-colored sl_N-invariant of the unknot. In the reduced homology one has $d_N = 0$, so

$$\mathcal{P}^{d_N,\mathrm{red}}_{\boxed{1}\boxed{2}} = \frac{(1+q^{2N+2}t^3)}{(1-q^4t^2)}.$$

3.3. **Antisymmetric projector, two strands.** The homology of the antisymmetric projector can be computed as follows. The HOMFLY-PT homology is an algebra with two even generators x_0, a_1 and two odd generators ξ_0, θ_1. The degrees are equal to:

$$\deg x_0 = q^2, \quad \deg a_1 = q^{-4}t^{-2}, \quad \deg \xi_0 = a^2t, \quad \deg \theta_1 = q^{-2}a^2t.$$

Therefore the Poincaré series of this homology equals:

$$\mathcal{P}_{\begin{smallmatrix}\boxed{1}\\ \boxed{2}\end{smallmatrix}} = \frac{(1+a^2t)(1+a^2q^{-2}t)}{(1-q^2)(1-q^{-4}t^{-2})}.$$

‡ \ t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
6				1		1			1		1		1		1		1		1		1	
4	1			1		2		2		2		1	1	1	1	1	1	1	1	1	1	1
2	1		1		1	1		2		3		3		3		2	1	2	1	2	1	2
0	1		1		2		2		2	1	1	2	1	3	1	3	1	3	1	2	2	2
-2					1		1		2		2		2	1	1	2	1	3	1	3	1	3
-4									1		1		2		2		2	1	1	2	1	3
-6													1		1		2		2		2	1
-8																	1		1		2	
-10																					1	

$\cdots$

$\cdots$

22	23	24	25	26	27	28	29	30	31	32	t / ‡
											6
1		1		1		1		1		1	4
1	1	1	1	1	1	1	1	1	1		2
1	2	1	2	1	2	1	2		1		0
2	2	2	2	2	2	1	1	1			-2
1	2	2	2	1	1	1					-4
1	2	1	1	1							-6
1	1	1									-8
1											-10

TABLE 3. The unreduced rational sl_3-homology of $T(3,16)$. See the caption of Table 1 for detailed explanations. The first half of the table depicts the stable part, and the second half the part diverging from $T(3,\infty)$.

The reduced Poincaré series equals

$$\mathcal{P}^{\mathrm{red}}_{\substack{\boxed{1}\\\boxed{2}}} = \frac{(1+a^2q^{-2}t)}{(1-q^{-4}t^{-2})}.$$

The differential is given by the formula

$$d_N(\xi_0) = x_0^N, \quad d_N(\theta_1) = Nx_0^{N-1}.$$

The homology is generated by x_0, a_1 and $N\xi_0 - x_0\theta_1$ modulo relation $x_0^{N-1} = 0$. The Poincaré series has the form:

$$\mathcal{P}^{d_N}_{\substack{\boxed{1}\\\boxed{2}}} = \frac{(1-q^{2N-2})(1+q^{2N}t)}{(1-q^2)(1-q^{-4}t^{-2})}.$$

The reduced Poincaré series equals:

$$\mathcal{P}^{d_N,\mathrm{red}}_{\substack{\boxed{1}\\\boxed{2}}} = \frac{(1+q^{2N-2}t)}{(1-q^{-4}t^{-2})}.$$

There is a natural embedding $\mathcal{H}^{\mathrm{alg}}(1,\infty) \hookrightarrow \mathcal{H}^{\mathrm{alg}}(2,\infty)$, and the Poincaré series of the quotient equals $-\mathcal{P}_{\substack{\boxed{1}\\\boxed{2}}}$. In other words, we have an identity

$$\mathcal{P}_{\boxed{1}\boxed{2}} + \mathcal{P}_{\substack{\boxed{1}\\\boxed{2}}} = \mathcal{P}_{\boxed{1}}.$$

Indeed, in $\mathcal{H}^{\mathrm{alg}}(1,\infty)$ we have generators ξ_0, x_0 and the differential $d_N(\xi_0) = x_0^N$. In $\mathcal{H}^{\mathrm{alg}}(2,\infty)$ we add the generators ξ_1, x_1 and the differential $d_N(\xi_1) = Nx_0^{N-1}x_1$. The quotient is spanned by the monomials containing either x_1 or ξ_1. If we introduce the formal variable $\theta_1 = \xi_1/x_1$, then the quotient would be spanned by all monomials divisible by x_1. On the other hand, $d_N(\theta_1) = Nx_0^{N-1}$.

3.4. **Homology of $(2, m)$-torus knots.** The unreduced sl_N-homology of $(2, m)$-torus knots was computed by Cautis in [4, Example 10.3.3], and can be reformulated as following:

Theorem 14. ([4]) *The Poincaré polynomial of the sl_N-homology of the $(2, 2k+1)$-torus knot is given by the following equation:*

$$\mathcal{P}(T(2,2k+1), sl_N) = \mathcal{P}^{d_N}_{\boxed{1}\boxed{2}} + q^{4k}t^{2k}\mathcal{P}^{d_N}_{\substack{\boxed{1}\\\boxed{2}}}.$$

The same decomposition holds for the reduced homology.

The reduced homology of $(2, 2k+1)$-torus knots is well known, see e.g. [7].

3.5. **Symmetric projector, three strands.** As above, the HOMFLY-PT homology of the symmetric projector is expected to coincide with the algebra $\mathcal{H}_3$, its Poincaré series equals (cf. [7]):

$$\mathcal{P}_{\boxed{1}\boxed{2}\boxed{3}} = \frac{(1+a^2t)(1+a^2q^2t^3)(1+a^2q^4t^5)}{(1-q^2)(1-q^4t^2)(1-q^6t^4)}, \quad \mathcal{P}^{\mathrm{red}}_{\boxed{1}\boxed{2}\boxed{3}} = \frac{(1+a^2q^2t^3)(1+a^2q^4t^5)}{(1-q^4t^2)(1-q^6t^4)}.$$

The unreduced sl_N-homology of the symmetric projector is given by Theorem 13:

$$\mathcal{P}^{d_N}_{\boxed{1}\boxed{2}\boxed{3}} = \frac{1}{(1-q^2)(1-q^4t^2)(1-q^6t^4)}(1 - q^{2N} - q^{2N+2}t^2 + q^{2N+4}(t^2+t^3-t^4) +$$

$$q^{2N+6}(t^4+t^5) - q^{4N+2}t^3 - q^{4N+4}t^5 - q^{4N+6}t^7 + q^{4N+10}(t^7+t^8) - q^{6N+6}t^8).$$

In the reduced homology we have $d_N = 0$ for $N > 2$ and $d_2(\xi_2) = x_1^2$. Therefore

$$\mathcal{P}^{d_N,\mathrm{red}}_{\ytableaushort{123}} = \frac{(1+q^{2N+2}t^3)(1+q^{2N+4}t^5)}{(1-q^4t^2)(1-q^6t^4)},\ N>2;\quad \mathcal{P}^{d_2,\mathrm{red}}_{\ytableaushort{123}} = \frac{(1-q^8t^4)(1+q^6t^3)}{(1-q^4t^2)(1-q^6t^4)}.$$

We also introduce a differential d_0 on the reduced homology by the formula $d_0(\xi_2) = x_1$. It is clear that

$$\mathcal{P}^{d_0,\mathrm{red}}_{\ytableaushort{123}} = \frac{(1+a^2q^2t^3)}{(1-q^6t^4)}.$$

3.6. **Antisymmetric projector, three strands.** The HOMFLY-PT homology of the antisymmetric projector is an algebra with two even generators x_0, a_1, a_2 and two odd generators $\xi_0, \theta_1, \theta_2$. The degrees are equal to:

$$\deg x_0 = q^2,\quad \deg a_1 = q^{-4}t^{-2},\quad \deg a_2 = q^{-6}t^{-2},$$
$$\deg \xi_0 = a^2t,\quad \deg \theta_1 = a^2q^{-2}t,\quad \deg\theta_2 = a^2q^{-4}t.$$

The Poincaré series equals:

$$\mathcal{P}_{\ytableaushort{1,2,3}} = \frac{(1+a^2t)(1+a^2q^{-2}t)(1+a^2q^{-4}t)}{(1-q^2)(1-q^{-4}t^{-2})(1-q^{-6}t^{-2})},\quad \mathcal{P}^{\mathrm{red}}_{\ytableaushort{1,2,3}} = \frac{(1+a^2q^{-2}t)(1+a^2q^{-4}t)}{(1-q^{-4}t^{-2})(1-q^{-6}t^{-2})}.$$

The differential is given by the formula

$$d_N(\xi_0) = x_0^N,\quad d_N(\theta_1) = Nx_0^{N-1},\quad d_N(\theta_2) = \binom{N}{2}x_0^{N-2}.$$

The homology has even generators x_0, a_1, a_2 and two odd generators $N\xi_0 - x_0\theta_1$, $\frac{N-1}{2}\theta_1 - x_0\theta_2$ modulo relation $x_0^{N-2} = 0$, so

$$\mathcal{P}^{d_N}_{\ytableaushort{1,2,3}} = \frac{(1-q^{2N-4})(1+q^{2N-2}t)(1+q^{2N}t)}{(1-q^2)(1-q^{-4}t^{-2})(1-q^{-6}t^{-2})}.$$

Note that for $N = 2$ this homology vanishes, as expected. In reduced homology $d_N = 0$, so

$$\mathcal{P}^{d_N,\mathrm{red}}_{\ytableaushort{1,2,3}} = \frac{(1+q^{2N-2}t)(1+q^{2N}t)}{(1-q^{-4}t^{-2})(1-q^{-6}t^{-2})}.$$

The differential d_0 on reduced homology is given by the formula $d_0(\theta_2) = a_1$, and

$$\mathcal{P}^{d_0,\mathrm{red}}_{\ytableaushort{1,2,3}} = \frac{(1+a^2q^{-2}t)}{(1-q^{-6}t^{-2})}.$$

3.7. **Hook-shaped projectors.** We formally introduce two power series $\mathcal{P}_{\ytableaushort{12,3}}$ and $\mathcal{P}_{\ytableaushort{13,2}}$ such that the following equations hold:

$$\mathcal{P}_{\ytableaushort{12}} = \mathcal{P}_{\ytableaushort{123}} + \mathcal{P}_{\ytableaushort{12,3}},\quad \mathcal{P}_{\ytableaushort{1,2}} = \mathcal{P}_{\ytableaushort{1,2,3}} + \mathcal{P}_{\ytableaushort{13,2}}.$$

One can check that for HOMFLY-PT homology these series are given by

$$\mathcal{P}_{\ytableaushort{12,3}} = \frac{(1+a^2t)(1+a^2q^{-2}t)(1+a^2q^2t^3)}{(1-q^2)(1-q^4t^2)(1-q^{-6}t^{-4})},$$

$$\mathcal{P}_{\ytableaushort{13,2}} = \frac{(1+a^2t)(1+a^2q^{-2}t)(1+a^2q^2t^3)}{(1-q^2)(1-q^{-4}t^{-2})(1-q^6t^2)},$$

$$\mathcal{P}^{\mathrm{red}}_{\ytableaushort{12,3}} = \frac{(1+a^2q^{-2}t)(1+a^2q^2t^3)}{(1-q^4t^2)(1-q^{-6}t^{-4})}, \quad \mathcal{P}^{\mathrm{red}}_{\ytableaushort{13,2}} = \frac{(1+a^2q^{-2}t)(1+a^2q^2t^3)}{(1-q^{-4}t^{-2})(1-q^6t^2)}.$$

For the unreduced sl_N-homology we have:

$$\mathcal{P}^{d_N}_{\ytableaushort{12,3}} = \frac{(1+q^{2N}t)(1-q^{2N-2}-q^{2N+2}t^2+q^{2N+4}t^2+q^{2N+4}t^3-q^{4N}t^3)}{(1-q^2)(1-q^4t^2)(1-q^{-6}t^{-4})} \tag{5}$$

$$\mathcal{P}^{d_N}_{\ytableaushort{13,2}} = \frac{(1+q^{2N}t)(1-q^{2N-2}-q^{2N+2}t^2+q^{2N+4}t^2+q^{2N+4}t^3-q^{4N}t^3)}{(1-q^2)(1-q^{-4}t^{-2})(1-q^6t^2)}. \tag{6}$$

For the reduced sl_N-homology we have

$$\mathcal{P}^{d_N,\mathrm{red}}_{\ytableaushort{12,3}} = \frac{(1+q^{2N-2}t)(1+q^{2N+2}t^3)}{(1-q^4t^2)(1-q^{-6}t^{-4})}, \quad \mathcal{P}^{d_N,\mathrm{red}}_{\ytableaushort{13,2}} = \frac{(1+q^{2N-2}t)(1+q^{2N+2}t^3)}{(1-q^{-4}t^{-2})(1-q^6t^2)} \ (N>2);$$

$$\mathcal{P}^{d_2,\mathrm{red}}_{\ytableaushort{12,3}} = \frac{(1-q^2)(1+q^6t^3)}{(1-q^4t^2)(1-q^{-6}t^{-4})}, \qquad \mathcal{P}^{d_2,\mathrm{red}}_{\ytableaushort{13,2}} = \frac{(1+q^2t)}{(1-q^{-4}t^{-2})} = \mathcal{P}^{d_2,\mathrm{red}}_{\ytableaushort{1,1}}.$$

These can also be interpreted as homology of d_N acting on some algebras.

For the diagram $\ytableaushort{12,3}$, the HOMFLY-PT homology is generated by $x_0, x_1, b_2, \xi_0, \xi_1, \theta_1$, and the differential has the form

$$d_N(\xi_0) = x_0^N, \quad d_N(\xi_1) = Nx_0^{N-1}x_1, \quad d_N(\theta_1) = Nx_0^{N-1} + \binom{N}{2}x_0^{N-2}x_1^2b_2.$$

Note that this differential can be obtained from the symmetric one: we set $\theta_1 = \xi_2/x_2$ and $b_2 = 1/x_2$, so

$$d_N(\theta_1) = \frac{1}{x_2}d_N(\xi_2) = \frac{1}{x_2}(Nx_0^{N-1}x_2 + \binom{N}{2}x_0^{N-2}x_1^2).$$

For the diagram $\ytableaushort{13,2}$, the HOMFLY-PT homology is generated by $x_0, a_1, x_2, \xi_0, \theta_1, \xi_2$ and the differential has the form

$$d_N(\xi_0) = x_0^{N-1}, \quad d_N(\theta_1) = Nx_0^{N-1}, \quad d_N(\xi_2) = \binom{N}{2}x_0^{N-2}x_2.$$

One can check that the Poincaré series for the homology of d_N agree with (5) and (6).

Finally, we define d_0 for the diagram $\ytableaushort{12,3}$ by the equation $d_0(\theta_1) = x_1b_2$, and for the diagram $\ytableaushort{12,3}$ by the equation $d_0(\xi_2) = x_2a_1$. One can check that

$$\mathcal{P}^{d_0,\mathrm{red}}_{\ytableaushort{12,3}} = \frac{(1-q^{-2}t^{-2})(1+a^2q^2t^3)}{(1-q^4t^2)(1-q^{-6}t^{-4})}, \quad \mathcal{P}^{d_0,\mathrm{red}}_{\ytableaushort{13,2}} = \frac{(1-q^2)(1+a^2q^{-2}t)}{(1-q^{-4}t^{-2})(1-q^6t^2)}.$$

3.8. **Homology of $(3, m)$-torus knots.** We summarize the known evidence for the Conjecture 2 in the following theorem.

Theorem 15. 1. *For the HOMFLY-PT homology, Conjecture 2 agrees with the "refined Chern-Simons invariants" defined in* [1]*. The conjecture is equivalent to the main conjecture of* [11].
2. *The conjecture holds for sl_2-homology.*
3. *The reduced homology of d_0 agrees with the hat version of the Heegaard-Floer homology of $(3, m)$-torus links after regrading $a = t^{-1}$ (cf.* [7, Section 3.8]*).*

Proof. The refined Chern-Simons invariants have been been presented as sums over SYT in [9], and the conjectural HOMFLY-PT homology of $(3, m)$-torus knots was first described in [7, Conjecture 6.8] and later reformulated in [11]. The sl_2-homology were computed by Turner in [20], and one can check his answers agree with the ones provided by Conjecture 2:

$$\mathcal{P}(T(3,3k+1),d_2) = \frac{1+q^2+q^4t^2+q^8t^3+q^{10}t^5+q^{12}t^5}{1-q^6t^4}+ \\ q^{6k}t^{4k}\frac{(1-q^6t^2)(1+q^4t)}{(1-q^4t^2)(1-q^{-6}t^{-4})}+q^{6k}t^{4k}\frac{1+q^4t}{1-q^{-4}t^{-2}},$$

$$\mathcal{P}(T(3,3k+2),d_2) = \frac{1+q^2+q^4t^2+q^8t^3+q^{10}t^5+q^{12}t^5}{1-q^6t^4}+ \\ q^{6k}t^{4k}\frac{(1-q^6t^2)(1+q^4t)}{(1-q^4t^2)(1-q^{-6}t^{-4})}+q^{6k+4}t^{4k+2}\frac{1+q^4t}{1-q^{-4}t^{-2}}.$$

Similar decomposition for sl_2 homology of $(3, m)$-torus knots was obtained in [15]. The Heegaard-Floer homology of torus knots is well known, see e.g [7, Section 6.12] for its description for $(3, m)$-torus knots. One can compare it with:

$$\mathcal{P}^{\mathrm{red}}(T(3,3k+1),d_0) = \frac{1+q^2t}{1-q^6t^4}+q^{6k}t^{4k}\frac{(1-q^{-2}t^{-2})(1+q^2t)}{(1-q^4t^2)(1-q^{-6}t^{-4})}+ \\ q^{6k}t^{4k}\frac{(1-q^2)(1+q^{-2}t^{-1})}{(1-q^{-4}t^{-2})(1-q^6t^2)}+q^{12k}t^{6k}\frac{1+q^{-2}t^{-1}}{1-q^{-6}t^{-2}},$$

$$\mathcal{P}^{\mathrm{red}}(T(3,3k+2),d_0) = \frac{1+q^2t}{1-q^6t^4}+q^{6k}t^{4k}\frac{(1-q^{-2}t^{-2})(1+q^2t)}{(1-q^4t^2)(1-q^{-6}t^{-4})}+ \\ q^{6k+4}t^{4k+2}\frac{(1-q^2)(1+q^{-2}t^{-1})}{(1-q^{-4}t^{-2})(1-q^6t^2)}+q^{12k+4}t^{6k+2}\frac{1+q^{-2}t^{-1}}{1-q^{-6}t^{-2}}.$$

□

For $N = 3$, we checked the conjecture up to the $(3, 83)$-torus knot.

4. DATA FOR STABLE sl_3-HOMOLOGY

In this section, the stable sl_3-homology of torus knots is compared against FoamHo calculations [14], which were conducted on a Xeon CPU E5-2620 with 128 GB RAM. Note that the quantum degree of the stable part of the sl_3-homology of an actual (n, m)-torus knot is shifted by $2(nm - n - m)$ relative to the homology of the (n, ∞)-torus knot.

4.1. $\mathbf{n=2.}$ The unreduced Poincaré series is given by (4):

$$\mathcal{P}_2^{\mathrm{alg}}(sl_3,\mathbb{Q})=\frac{1-q^6-q^8t^2+q^{10}t^2+q^{10}t^3-q^{14}t^3}{(1-q^2)(1-q^4t^2)}.$$

The reduced Poincaré series equals:

$$\mathcal{P}_2^{\mathrm{alg,red}}(sl_3,\mathbb{Q})=\frac{1+q^8t^3}{1-q^4t^2}.$$

All of homology is stable, e.g. the first divergence when comparing with the $(2,201)$-torus knot occurs at homological degree 202.

4.2. $\mathbf{n=3.}$ The unreduced Poincaré series is given by Theorem 13:

$$\mathcal{P}_3^{\mathrm{alg}}(sl_3,\mathbb{Q})(1-q^2)(1-q^4t^2)(1-q^6t^4)=$$
$$1-q^6-q^8t^2+q^{10}t^2+q^{10}t^3-q^{14}t^3-q^{10}t^4+q^{12}t^4+q^{12}t^5-q^{16}t^5-q^{18}t^7+q^{22}t^7+q^{22}t^8-q^{24}t^8.$$

The reduced Poincaré series equals:

$$\mathcal{P}_3^{\mathrm{alg,red}}(sl_3,\mathbb{Q})=\frac{(1+q^8t^3)(1+q^{10}t^5)}{(1-q^4t^2)(1-q^6t^4)}.$$

Compared with the rational homology of the $(3,83)$-torus knot (calculation took 6 hours and 1.5 GB RAM), the first divergence is at $q^{168}t^{112}$, both for unreduced and for reduced homology.

4.3. $\mathbf{n=4.}$ The computations of the Koszul homology were done with Singular [6], a computer algebra system. The unreduced Poincaré series equals:

$$\mathcal{P}_4^{\mathrm{alg}}(sl_3,\mathbb{Q})(1-q^2)(1-q^4t^2)(1-q^6t^4)(1-q^8t^6)=$$
$$1-q^6-q^8t^2+q^1t^2+q^{10}t^3-q^{14}t^3-q^{10}t^4+q^{12}t^4+q^{12}t^5-q^{16}t^5-q^{12}t^6+q^{14}t^6+q^{14}t^7-2q^{18}t^7+$$
$$q^{22}t^7+q^{22}t^8-q^{24}t^8-q^{20}t^9+q^{24}t^9+q^{24}t^{10}-q^{26}t^{10}-q^{22}t^{11}+q^{26}t^{11}+q^{26}t^{12}-q^{28}t^{12}-q^{30}t^{14}+q^{36}t^{14}$$

The reduced Poincaré series equals:

$$\mathcal{P}_4^{\mathrm{alg,red}}(sl_3,\mathbb{Q})=\frac{(1+q^8t^3)(1+q^{10}t^5)(1-q^{12}t^6)}{(1-q^4t^2)(1-q^6t^4)(1-q^8t^6)}.$$

Compared with the rational homology of the $(4,35)$-torus knot (computation took 12 days and 25 GB RAM), the first divergence is at $q^{72}t^{54}$, both for unreduced and for reduced homology.

4.4. $\mathbf{n=5.}$ The unreduced Poincaré series (again computed by Singular) equals:

$$\mathcal{P}_5^{\mathrm{alg}}(sl_3,\mathbb{Q})(1-q^2)(1-q^4t^2)(1-q^6t^4)(1-q^8t^6)(1-q^{10}t^8)=$$
$$1-q^6-q^8t^2+q^{10}t^2+q^{10}t^3-q^{14}t^3-q^{10}t^4+q^{12}t^4+q^{12}t^5-q^{16}t^5-q^{12}t^6+q^{14}t^6+q^{14}t^7-2q^{18}t^7+$$
$$+q^{22}t^7-q^{14}t^8+q^{16}t^8+q^{22}t^8-q^{24}t^8+q^{16}t^9-2q^{20}t^9+q^{24}t^9+q^{24}t^{10}-q^{26}t^{10}-2q^{22}t^{11}+2q^{26}t^{11}+$$
$$2q^{26}t^{12}-2q^{28}t^{12}-q^{24}t^{13}+q^{28}t^{13}+q^{26}t^{14}-2q^{30}t^{14}+q^{36}t^{14}-q^{28}t^{15}+q^{30}t^{15}+q^{32}t^{15}-q^{34}t^{15}+q^{30}t^{16}$$
$$-q^{32}t^{16}-q^{34}t^{16}+q^{38}t^{16}+q^{34}t^{17}-q^{36}t^{17}-q^{36}t^{18}+2q^{40}t^{18}-q^{42}t^{18}+q^{36}t^{19}-q^{38}t^{19}+q^{40}t^{19}-q^{42}t^{19}$$
$$-q^{38}t^{20}+2q^{42}t^{20}-q^{44}t^{20}+q^{42}t^{21}-q^{44}t^{21}+q^{44}t^{22}-q^{46}t^{22}-q^{46}t^{23}+q^{50}t^{23}.$$

The reduced Poincaré series equals:

$$\mathcal{P}_5^{\mathrm{alg,red}}(sl_3, \mathbb{Q}) = \frac{(1+q^8t^3)(1+q^{10}t^5)(1-q^{12}t^6-q^{14}t^8+q^{18}t^{10}+q^{18}t^{11}-q^{26}t^{15})}{(1-q^4t^2)(1-q^6t^4)(1-q^8t^6)(1-q^{10}t^8)}$$

Compared with the rational homology of the $(5,14)$-torus knot (computation took 15 days and 33 GB RAM), the first divergence is at $q^{30}t^{24}$, both for unreduced and for reduced homology.

REFERENCES

[1] M. Aganagic, S. Shakirov. *Refined Chern-Simons theory and knot homology*. String-Math 2011, 3–31, Proc. Sympos. Pure Math., **85**, Amer. Math. Soc., Providence, RI, 2012.

[2] D. Bar-Natan. *Fast Khovanov Homology Computations*. J. Knot Theory Ramifications **16** (2007), no. 3, 243–255.

[3] D. Bar-Natan, S. Morrison. *The Knot Atlas*. `http://katlas.org`.

[4] S. Cautis. *Clasp technology to knot homology via the affine Grassmannian*. arXiv:1207.2074.

[5] B. Cooper, M. Hogancamp. *An Exceptional Collection For Khovanov Homology*. arXiv:1209.1002.

[6] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann. SINGULAR *3-1-3 — A computer algebra system for polynomial computations*. `http://www.singular.uni-kl.de` (2011).

[7] N. Dunfield, S. Gukov, J. Rasmussen. *The superpolynomial for knot homologies*. Experiment. Math. **15** (2006), no. 2, 129–159.

[8] E. Gorsky, S. Gukov, M. Stošić. *Quadruply-graded colored homology of knots*. arXiv:1304.3481.

[9] E. Gorsky, A. Negut. *Refined knot invariants and Hilbert schemes*. arXiv:1304.3328.

[10] E. Gorsky, A. Oblomkov, V. Shende. *On stable Khovanov homology of torus knots*. Experiment. Math. **22** (2013), no. 3, 265–281.

[11] E. Gorsky, A. Oblomkov, J. Rasmussen, V. Shende. *Torus knots and the rational DAHA*. arXiv:1207.4523.

[12] V. Jones, M. Rosso. *On the invariants of torus knots derived from quantum groups*. J. Knot Theory Ramifications **2** (1993), no. 1, 97–112.

[13] L. Lewark. *$\mathfrak{sl}_3$-foam homology calculations*. Algebr. Geom. Topol. **13** (2013), no. 6, 3661–3686.

[14] L. Lewark. FOAMHO*, an $\mathfrak{sl}_3$-homology calculator*. `http://www.maths.dur.ac.uk/~vxhn54/foamho.html` (2012).

[15] A. Oblomkov, J. Rasmussen, unpublished, 2011.

[16] J. Przytycki, R. Sazdanović. *Torsion in Khovanov homology of semi-adequate links*. arXiv:1210.5254.

[17] D. Rose. *A Categorification of Quantum sl_3 Projectors and the sl_3 Reshetikhin-Turaev Invariant of Tangles*. arXiv:1109.1745.

[18] L. Rozansky. *An infinite torus braid yields a categorified Jones-Wenzl projector*. arXiv:1005.3266.

[19] A. Shumakovitch. *Torsion of the Khovanov homology*. arXiv:math/0405474.

[20] P. Turner. *A spectral sequence for Khovanov homology with an application to $(3,q)$–torus links*. Algebr. Geom. Topol. **8** (2008) 869–884.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY
2990 BROADWAY NEW YORK, NY 10027 USA
E-mail address: `egorsky@math.columbia.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY
SCIENCE LABS, SOUTH ROAD, DURHAM DH1 3LE, UK
E-mail address: `lukas.lewark@durham.ac.uk`